A SIMPLE DERIVATION OF A FORMULA OF FURSTENBERG AND TZKONI

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ABSTRACT

It is shown that an integral geometric formula concerning n-dimensional ellipsoids, due to Furstenberg and Tzkoni, is a corollary of a classical formula of Blaschke and Petkantschin.

Let E denote an n-dimensional ellipsoid centred at the origin of R^n and (for $k = 1,\cdots,n-1$) let F_k^n denote the Grassmannian manifold of k-dimensional subspaces ξ of R^n. Furstenberg and Tzkoni [2] give two derivations, both somewhat "high-powered", of the integral geometric formula

$$c_{n,k}V_n(E)^k = \int_{F_k^n} V_k(E \cap \xi)^n dm(\xi) \tag{1}$$

where $V_n(E)$ denotes the n-dimensional volume of E, $dm(\xi)$ is the normalized rotation invariant density element in F_k^n, $V_k(E \cap \xi)$ denotes the k-dimensional volume of the ellipsoidal section $E \cap \xi$, and $c_{n,k}$ is a constant depending only on n and k. (Taking E to be the unit ball B, we find

$$c_{n,k} = \left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^k \Big/ \left\{\frac{k}{2}\Gamma\left(\frac{k}{2}\right)\right\}^n .) \tag{2}$$

There now follows a simple alternative derivation of (1). Let $x_1,\cdots,x_k$ be points of R^n such that the corresponding vectors $X_1,\cdots,X_k$ from the origin are linearly ndependent, let ξ be the k-dimensional subspace spanned by these vectors, and et $x_1^k,\cdots,x_k^k$ "represent" $x_1,\cdots,x_k$, respectively, with respect to k-dimensional cartesian coordinates in ξ (a complete specification would involve the notion o

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fibre bundles). Blaschke [1] and Petkantschin [4] derived the fundamental integral geometric density relation

$$\prod_{1}^{k} dx_i = \nabla_k^{n-k} d\bar{m}(\xi) \prod_{i=1}^{k} dx_i^k \tag{3}$$

where dx_i, dx_i^k denote n- and k-dimensional Lebesgue volume elements, respectively, $d\bar{m}(\xi)$ denotes the (un-normalized) invariant density element in F_k^n, and ∇_k is the k-dimensional volume of the parallelotope with edges $X_1, \cdots, X_k$. For a simple proof of (3), see [3, §§2, 3]. Thus

$$\begin{aligned} V_n(E)^k &= \int \cdots \int_{E^k} \prod_{1}^{k} dx_i \\ &= \int_{F_k^n} \int \cdots \int_{(E\cap\xi)^k} \nabla_k^{n-k} \prod_{i=1}^{k} dx_i^k d\bar{m}(\xi) \end{aligned} \tag{4}$$

which, by affine transformation within ξ,

$$\begin{aligned} &= \int_{F_k^n} \{V_k(E\cap\xi)/V_k(B\cap\xi)\}^n \\ &\qquad \int \cdots \int_{(B\cap\xi)^k} \nabla_k^{n-k} \prod_{i=1}^{k} dx_i^k d\bar{m}(\xi) \\ &= (c_{n,k})^{-1} \int_{F_k^n} V_k(E\cap\xi)^n dm(\xi). \end{aligned}$$

Q.E.D

Furstenberg and Tzkoni indicate that, when $k = 1$, (1) holds for any symmetric star-shaped body in R^n, and ask whether, when $k > 1$, (1) is valid for more general bodies than the ellipsoid. The present proof suggests a negative answer.

As a corollary of (4), we have

$$\int \cdots \int_{B^n} \nabla_n^t \prod_{1}^{n} dx_i = \left\{\pi^{n/2}/\Gamma\left(\frac{n+t}{2}+1\right)\right\}^n \prod_{j=1}^{n} \left\{\Gamma\left(\frac{j+t}{2}\right)/\Gamma\left(\frac{j}{2}\right)\right\} \tag{5}$$

$$(t = 0, 1, 2, \cdots).$$

The values of many further "isotropic" multiple integrals like (5) are to be found in [3]. The derivations stem from (3) and the companion Blaschke-Petkantschin formula

$$\prod_{1}^{k} dx_i = \{(k-1)!\Delta_{k-1}\}^{n-k+1} d\bar{M}(\eta) \prod_{i=1}^{k} dx_i^{k-1} \quad (k = 2, \cdots, n), \tag{6}$$

where η is the $(k-1)$-dimensional flat containing $x_1, \cdots, x_k$, $d\bar{M}(\eta)$ is the corresponding invariant density element, x_i^{k-1} represents x_i within η, and Δ_{k-1} is the $(k-1)$-dimensional volume of the simplex convex hull of $x_1, \cdots, x_k$ (see [4, p. 275]).

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